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Application of Fermat's Principle to Magnetic Spectrometers

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The principle of Fermat (or of Maupertuis) contains implicitly the entirety of geometrical optics. It is used in the present paper to find the trajectories of charged particles passing through magnetic systems possessing a plane of radial symmetry. A general formula is given for the calculation of the aberrations at image points. The method is illustrated in its application to certain systems of special interest in which specific second order aberrations cancel. The method outlined herein is believed to possess the advantage of relative simplicity and directness when compared to more general formulations.

I. INTRODUCTION

To calculate a particle trajectory in a static electromagnetic field, many authors have extensively used the Hamiltonian method. One may also use an equivalent procedure, which is an application of the Fermat (or Maupertuis) principle: the real trajectory followed by the particle is such that some function associated with it is an extremum. This last method was used by Cotte, Glaser, and Sturrock, among many others.

We intend to apply this method to the particular case of magnetic spectrometers or, more generally, magnet systems possessing a radial symmetry plane. We first develop the method in Sec. II and derive a general expression for the optical length associated with a trajectory. This expression permits rapid obtainment of the differential equation of the trajectory. For the clarity of the text, we reproduce some already well known applications.

In Sec. III, we examine the problem of aberrations, limiting ourselves to those of second order. We derive a formula which permits the calculation of any specific aberration term at the image point. The various expressions are presented in a table.

In Sec. IV we show, with specific examples, how the method permits design of rapidly magnetic spectrometers where some specific aberrations do cancel.

II. FERMAT'S PRINCIPLE APPLIED TO MAGNETIC SPECTROMETERS

1. Notations

One considers a magnet, or a magnetic system, where the magnetic field lines exhibit a radial symmetry plane. One supposes that the particle has a negative charge $-e/c$, relativistic mass $m$, velocity $v$, and momentum $p=mv$. A possible trajectory in the symmetry plane is taken as a reference trajectory and is called the optical axis. This optical axis is conveniently chosen; for instance, inside a circular spectrometer, it coincides with the magnet's mean radius circle. Particles following the optical axis have a momentum $p_0$; any other particle has a momentum $p=p_0+\Delta p$.

Streib's notations are used. A point $P$ in space is defined by the curvilinear coordinates $(t,r,z)$ (Fig. 1). The relation between the value $H(t)$ of the field along the optical axis and the curvature $h(t)=1/R(t)$ of this axis is

$$p_0 h=-H.$$  \hspace{1cm} (1)

The orientation convention is such that $t$, $r$, $z$, and $h$ are positive in the case of Fig. 1. The magnetic field is perpendicular to the plane of symmetry, that is,

$$B_z(t,r,0)=B_z(t,r,0)=0.$$  \hspace{1cm} (2)

We develop the $z$ component of the magnetic field as a power series in $rh$, keeping terms up to the second order; this yields

$$B_z(t,r,0)=H \left(1+\frac{nh}{2} \right),$$  \hspace{1cm} (2)

where $H$, $h$, $n$, $m$ are functions of $t$ only. These variables usually have constant values over a wide range of the

![Fig. 1. Curvilinear coordinates are defined relatively to the optical axis, located in the symmetry plane. $M(t,0,0)$ is the orthogonal projection on that axis of any point $P(t,r,z)$, where $t$ is the abscissa along the axis, $r$ and $z$ the radial and transverse coordinates. The particle has a negative charge.


$^2$ Many authors use $\theta$ instead of $m/2$.}
variable $t$ inside the magnet; outside the magnet, they take the value zero.

Maxwell relations and the symmetry condition permit us to obtain the general components of $B$ (here, and later, all derivatives, when not specified, are with respect to $t$).

$$B_r(t,r,z) = H(nhz + mhr)$$

$$B_z(t,r,z) = H'(z) + [hH + (Hn)'r]z.$$ 

2. Fermat's Principle

The “generalized momentum” is given by

$$P = mv - eA,$$  

where $A$ is a vector potential defining the magnetic field.

All corpuscular optics may then be deduced from the following well-known theorem:

For any trajectory between two given points in space, the following line integral, which is called the “optical length,”

$$T = \int P \cdot ds$$  

is extremal. The running variable $s$ refers to actual distance measured along the trajectory.

It has been shown by Sturrock (p. 60 of Ref. 4) that this optical length is always a true minimum.

3. Physical Interpretation

One considers two possible trajectories, 1 and 2, between A and B (Fig. 2). One gives an interpretation of the quantity $T(2) - T(1)$. The line integral of $P$ is the sum of two terms. The integral of $mv$ is $p = me$ times the length of the trajectory; this gives a term $p(s_2 - s_1)$. Then, owing to Stokes theorem, the difference in the integral of $A$ along the two trajectories is equal to the flux of the magnetic field through the surface. Finally, one may write

$$\Delta T = T(2) - T(1) = p(s_2 - s_1) + e\phi,$$  

where $\phi$ is the flux of the magnetic field through the surface A1B2 oriented in that sense.

4. Evaluation of $T$

One supposes that the trajectory $\Xi$ is given by $r(t)$ and $z(t)$, and one evaluates the optical length $T$ when these functions, and their derivatives, are small;

$$hr \ll 1 \quad hz \ll 1$$

$$r' = dr/dt \ll 1 \quad z' = dz/dt \ll 1$$

$$\Delta p/pB \ll 1.$$
$T$ is calculated up to third order terms only. The variation of $T$ between two neighboring points $(t)$ and $(t+dt)$ is

$$\Delta T = \frac{\partial T}{\partial \mathbf{ds}} - \mathbf{A} \cdot \mathbf{ds}.$$ 

Calculations are outlined briefly. The same method as Glaser's is used basically. One notes first that

$$\mathbf{ds} = \sqrt{[1 - hr]^2 + r'^2 + z'^2] \mathbf{dt}}.$$ 

In our coordinates, the curl $\mathbf{B}$ is given by

$$B_t = -Az - A_r,$$

$$B_r = -(1-rh)A_t - A_z,$$

$$B_z = -(1-rh)A_r - A_t.$$ 

The solution of (6) satisfying conditions (7) is (neglecting higher order terms)

$$A_r = -\frac{H}{hr} - \frac{r'^2 + z'^2}{h}$$

$$A_z = 0$$

$$A_t = \frac{1}{h} \left[ -hr + (1-n) \frac{h^2r^2}{2} + \frac{n}{2} \right]$$

$$+ \frac{h^2r^2}{3} + \frac{(m-n)hr^2}{2} + \frac{(n-m)hr^2}{2}.$$ 

Therefore,

$$\mathbf{ds} = \mathbf{ds} - A \mathbf{ds} + \mathbf{A} \mathbf{ds} + A \mathbf{r} \left[ 1 - rh \right] \mathbf{dt}$$

$$= \mathbf{ds} - A \mathbf{ds} + \mathbf{A} \mathbf{ds} + \mathbf{A} \mathbf{r} \left[ 1 - rh \right] \mathbf{dt}$$

Taking into account the third order terms in $\mathbf{F}$, one obtains in the symmetry plane

$$T'' + \left[ 1 - n \right] h^2r + \frac{\Delta \mathbf{p}}{\mathbf{p}_0} h^2 = \left[ \frac{\Delta \mathbf{p}}{\mathbf{p}_0} + h^2.r ight]$$

and in the transverse plane

$$\varphi'' + nh^2z = - \left[ \frac{\Delta \mathbf{p}}{\mathbf{p}_0} + hr \right] \varphi'' - (hr)z'$$

Finally the evaluation of $T$ between two arbitrary points $A$ and $B$ is given by

$$T = \mathbf{p}_0 \int_A^B F \mathbf{dt},$$ 

where $F$ is the so-called "variational function." $F$ and $T$ can be expressed as sums of functions of order zero, one, etc., of the variables $hr, hs, r', z'$, and $\Delta \mathbf{p}/\mathbf{p}_0$. Up to the third order one has

$$F = F_0 + F_1 + F_2 + F_3,$$

$$T = \mathbf{p}_0 (T_0 + T_1 + T_2 + T_3),$$

where

$$F_0 = \frac{\Delta \mathbf{p}}{\mathbf{p}_0},$$

$$F_3 = \frac{\Delta \mathbf{p}}{\mathbf{p}_0} + hr \left[ \frac{\Delta \mathbf{p}}{\mathbf{p}_0} + h^2.r ight] + \frac{h^2 \mathbf{r}^2}{3}$$

$$T_n = \int_A^B \mathbf{F} \mathbf{dt}.$$ 

5. Derivation of the Trajectory Equations, Applications

The differential equations of the trajectories are deduced from the Euler equations,

$$\frac{d}{dt} \left( \frac{\partial F}{\partial \varphi'} \right) - \frac{\partial F}{\partial \varphi} = 0,$$

$$\frac{d}{dt} \left( \frac{\partial F}{\partial r'} \right) - \frac{\partial F}{\partial r} = 0.$$ 

To second order in $F$ one finds

$$r'' + \left[ 1 - n \right] h^2r + \frac{\Delta \mathbf{p}}{\mathbf{p}_0} = 0$$

$$z'' + nh^2z = 0.$$ 

This is called the paraxial or Gaussian approximation. $T_3$ is constant along the various paraxial trajectories between two conjugate points.

Streib has obtained equivalent equations by a different method.
The first order optics of magnetic spectrometers and beam transport systems may be obtained from Eqs. (13) and (14). For the clarity of the text some well known consequences are reproduced (see, for instance, Refs. 2 and 9).

Let us suppose $\Delta \phi = 0$. Then Eq. (13) becomes

$$r' + (1-n)h^2 r = 0.$$  \hspace{1cm} (17)

Among the solutions to this equation in $r(t)$, we call $C_r(t)$ those which satisfy the conditions $r(0) = 1$ and $r'(0) = 0$; and $S_r(t)$ those which satisfy the conditions $r(0) = 0$ and $r'(0) = 1$.

$$C_r'' + (1-n)h^2 C_r = 0 \quad C_r(0) = 1 \quad C_r'(0) = 0$$

$$S_r'' + (1-n)h^2 S_r = 0 \quad S_r(0) = 0 \quad S_r'(0) = 1. \hspace{1cm} (18)$$

In the $z$ plane, one may define analogously the solutions $C_z(t)$ and $S_z(t)$ of Eq. (14). When $\Delta \phi = 0$, any real ray is, in the paraxial approximation, a linear combination of the four rays so defined.

It is interesting to note that the derivative of the quantity $CS' - SC''$ is $CS'' - SC'$, which is zero according to Eq. (17). Therefore the determinant $CS' - SC''$ is constant and equal to its value for $t=0$, that is, according to conditions (18), unity. For conjugate points, this theorem takes a particular form, known as the Lagrange-Helmholtz relation: The product of linear and angular magnifications is equal to 1.

If $\Delta \phi \neq 0$, then in the radial plane the equation of the trajectory is a solution of (13). Let us define $D(t)$ as the solution in the particular case $\Delta \phi / \phi_0 = -1$.

$$D'' + (1-n)h^2 D = h,$$

with the boundary conditions

$$D(0) = 0, \quad D'(0) = 0.$$

Using the Lagrange method of varying parameters, we may express $D(t)$ as

$$D(t) = S_r(t) \int_0^t C_r(u) h(u) du - C_r(t) \int_0^t S_r(u) h(u) du. \hspace{1cm} (19)$$

Generalizing now the preceding statement, we may say that any real trajectory is, in the paraxial approximation, a linear combination of the five particular ones: $C_r, S_r, D, C_z, S_z$.

The transfer matrix along a trajectory is given by

$$\begin{bmatrix}
    r_2 \\
    r'_2 \\
    \Delta \phi / \phi_0
\end{bmatrix} = 
\begin{bmatrix}
    C_r & S_r & -D \\
    C'_r & S'_r & -D' \\
    0 & 0 & 1
\end{bmatrix} 
\begin{bmatrix}
    r_1 \\
    r'_1 \\
    \Delta \phi / \phi_0
\end{bmatrix}$$

$$\begin{bmatrix}
    z_2 \\
    z'_2
\end{bmatrix} = 
\begin{bmatrix}
    C_z & S_z \\
    C'_z & S'_z
\end{bmatrix} 
\begin{bmatrix}
    z_1 \\
    z'_1
\end{bmatrix}.$$

Conjugate points in the symmetry plane correspond to the condition $S_r(t_i) = 0$, where $t_i$ is the abscissa of the image point. Then the various elements of the transfer matrix may be interpreted as

- Linear magnification
- Angular magnification
- Dispersion
- Angular dispersion

$$\begin{bmatrix}
    0 \\
    -(\text{Focal length})^{-1} \\
    0 \\
    0 \\
    1
\end{bmatrix}$$

III. SECOND ORDER ABERRATIONS

1. General Method

We try to solve Eqs. (15) and (16). As we are interested in the second order terms only, we may make, on the right hand side, the substitutions

$$r = r_0 C_r + r'_0 S_r - D, \hspace{1cm} \Delta \phi / \phi_0$$

$$z = z_0 C_z + z'_0 S_z, \hspace{1cm} \Delta \phi / \phi_0$$

We obtain therefore a polynomial in $r_0^2, r_0, r'_0$ etc. . . . of degree 2, whose coefficients are functions of $t$ only. The same method of varying parameters, used to derive the $D$ function [Eq. (19)], can be applied to solve this equation. It leads to the so-called "Streib's coefficients" which are tabulated.\textsuperscript{5,7}

2. A Useful Theoretical Expression

One may prove the following formula: At the image point, all second order linear aberrations are given in the radial plane by

$$r - r_0 = -M_r \left( \frac{\partial T_3}{\partial r_0} \right), \hspace{1cm} (21)$$

and in the transverse plane by

$$z - z_0 = -M_z \left( \frac{\partial T_3}{\partial z_0} \right),$$

where $M_r, z_0$ are the linear magnifications, $r_0, z_0$ the values of $r$ and $z$ at the object point, and $r, z$ the paraxial values of $r$ and $z$ at the image point.
Let us give a direct demonstration. One considers trajectories in the symmetry plane \((z=0)\). The object is \(AA'\), and its paraxial image is \(BB'\). A real ray starting from \(A'\) with an initial \(r'_0\) crosses the image plane through a point \(B'\) different from \(B'_0\). The aberration at \(B\) is the distance \(B'B''\).

We consider now two paraxial rays \(g\) and \(g'\) starting from \(A'\) with values of \(r'_0\), \(r'_0\), and \(r'_0 + \Delta r'_0\), respectively. They pass through the paraxial image point \(B'_0\). Let us evaluate the optical length \(T = \int P \, ds\) along one of these two rays, say \(g\). We consider now two paraxial rays \(g\) and \(g'\) starting from \(A'\) with values of \(r'_0\), \(r'_0\), and \(r'_0 + \Delta r'_0\), respectively. They pass through the paraxial image point \(B'_0\). Let us evaluate the optical length \(T = \int P \, ds\) along one of these two rays, say \(g\). It is equal to the optical length evaluated along the real ray \(\Sigma\) plus an increment \(\delta T\). From a well known formula of the variational calculus (which depends on the fact that along \(\Sigma\), the optical length is extremum), one obtains this increment:

\[
\delta T = [\mathbf{P} \cdot \Delta \mathbf{r}]_0 - [\mathbf{P} \cdot \Delta \mathbf{r}]_0 = [\mathbf{P} \cdot \mathbf{B}'_0 \cdot \mathbf{B}].
\]  

(22)

If we evaluate the optical length along \(g'\) we find a different value of \(\delta T\). The difference between the two optical lengths is

\[
\Delta T = \text{Optical length along } g' - \text{Optical length along } g
\]

= Variation at \(B'_0\) of \([\mathbf{P} \cdot \mathbf{B}'_0 \cdot \mathbf{B}].\)

Using Eq. (3) we find

\[
\Delta T = p_0 \cdot B'_0 \cdot D \cdot r'_0 = -p_0 (r - r_0) \Delta r'_0,
\]

(23a)

where \(\Delta r'_0\) is the angle between the two Gaussian rays at \(B'_0\).

One may also use the Eqs. (9), (10), and (11) to derive another expression for \(\Delta T\). \(T_0\) and \(T_1\) are clearly the same for any curve starting at \(A\) and finishing at \(B\); the paraxial trajectories are such that between two conjugate points, \(T_0\) is the same. Therefore the variation \(\Delta T\) of optical length along a paraxial trajectory is, neglecting higher order terms,

\[
\Delta T = p_0 \Delta T_0.
\]

(23b)

Finally, comparing (23a) and (23b), one finds

\[
r - r_0 = -\frac{\Delta T_0}{\Delta r'_0}.
\]

By comparing this with formula (21), one finds the Lagrange-Helmholtz relation,

\[
\frac{\Delta r'_0}{\Delta r} \times M_s = 1.
\]

This demonstration of Eq. (21) is not rigorous because in formula (22) \(\Delta r\) is supposed to be an infinitely small quantity while in fact \(B'B''\) is a small but finite quantity. However, Glaser\(^2\) noting that both sides of Eq. (15) and (16) are derived rigorously from the same Euler equation, shows that one may go a step further and derive general formulae for the aberrations, a particular case of which is given by formula (21). Sturrock\(^3\),\(^4\) has also given a direct demonstration, which is similar to ours but more general.

### 3. Development of Formula (21)

In Eq. (11) let us replace \(r, r', \ldots\) by their paraxial approximations (20) and derive the expression relatively...
to \( r_0' \) and \( z_0' \). The results are given in Table I, which permits calculating from Eq. (21) any specific aberration coefficient at the image point. For instance, the coefficient of \( (r_0' \Delta \rho/\rho_0) \) in \( r-r_0 \) is

\[
\left( \frac{r/r_0'}{\rho_0} \right) = -M \int_{\text{image}} \left( S_{r''} (1 - hD) - 2hS_{r'} S_{r'} D' \right)
- 2 \left( \frac{m}{2} - n \right) h^2 S_{r'} D' dt. \tag{24}
\]

In case of constant field \([h \text{ and } n \text{ constant}, \text{ and } h(1-n)\neq0]\), the following identity may be useful:

\[
\int_a^b \left( \frac{\Delta \rho}{\rho_0} + hr \right) \frac{r^2}{2} dt = -\frac{1}{2(1-n)} \int_a^b \frac{\Delta \rho}{\rho_0} \frac{r^2}{2} dt
- \frac{1}{6h(1-n)} [r'']_b^a. \tag{25}
\]

It can be shown that the aberration coefficients given in Table I are equivalent to Streib's when his formulae are evaluated at the image point.

For an application of our formalism, see Appendix 2.

IV. DESIGN OF ABERRATIONLESS MAGNETIC SPECTROMETERS

In this section we ignore dispersive (or chromatic) terms, that is we put \( \Delta \rho = 0 \). Equation (21) leads to the following interesting application.

Consider only trajectories in the symmetry plane \((z=0)\) and let A and B be two conjugate points. It is a consequence of Eq. (21) that if the optical path is constant (to third order terms) for any paraxial ray between A and B, all second order aberrations in the symmetry plane are zero at B.

A similar statement holds for the transverse plane. We shall give two examples of applications.

1. The "Magic Magnet" as an Application of Fermat's Principle

Consider a constant-gradient magnetic spectrometer with a deflection angle \( \alpha \), mean radius \( R \), and normal entrance and exit faces (Fig. 5). Any radial ray in the magnet has a Gaussian trajectory given by

\[
r = A \sin k \left( \frac{t-t_0}{R} \right), \tag{26}
\]

where \( A \) and \( t_0 \) are constant parameters, and \( k^2 = 1-n \).

For a double focusing spectrometer, \( n = \frac{1}{2} \), therefore \( m = \frac{3}{2} \) and \( \alpha = 2\pi/3 = 120^\circ \ldots \). It is the so-called "magic magnet" discovered by Sakai and recently constructed and studied by Penner at the National Bureau of Standards. Whatever the dimension and position of the object, there will be no second order radial aberrations at the image for trajectories lying in the symmetry plane.

Another interesting case is the \( n = 0 \) quasi-uniform field magnet with

\[
\begin{align*}
1 - 3n + m &= 0 \\
3k\alpha &= 2\pi.
\end{align*}
\]

2. Modifications of Optical Properties at the Boundaries of a Uniform Field

One takes a magnet \( M \) with uniform field \((n=m=0)\) and supposes that the entrance and exit faces are normal.

to the central ray, but have a curvature \( \rho \) (Fig. 6). One writes

\[
T_s = \int_{\text{magnet } M} \frac{hr'^3}{2} dt + \text{contributions at entrance and exit.}
\]

Consider another magnet \( M' \), identical but without any curvature, and consider the same trajectory (identical in paraxial approximation). For this magnet

\[
T_s' = \int_{\text{magnet } M'} \frac{hr'^3}{2} dt.
\]

Thus, contributions at entrance and exit of magnet \( M = T_s - T_s' \).

This difference may be evaluated by use of formula (5); with neglect of 4th order terms only the flux parts are different, so that one may write, for the entrance,

\[
(T_s - T_s')_{t_e} = \frac{-1}{\rho_0} \times \text{missing flux through } I_0 II'.
\]

\[
= -h \frac{r^3_0}{6\rho_1}.
\]

(28)

Let us consider the case of symmetric object and image. Then, according to Eq. (25),

\[
\int \frac{hr'^2}{2} dt = \frac{1}{6h} \left[ \frac{R}{r^3} \right] = \frac{R}{3}.
\]

Using Eq. (28),

Contributions at \( M \) and \( N = -h \frac{r^3_0}{6\rho_1} - h \frac{r^3_0}{6\rho_2} \).

The sum of these two gives

\[
T_s = -\frac{R}{3} \frac{hr'^3}{6} \left[ \frac{1}{\rho_1} + \frac{1}{\rho_2} \right].
\]

If \( 2\theta \) is the deflection angle, then

\[
T_s = \frac{R}{6} \left[ 2 - R \cot^2 \theta \left( \frac{1}{\rho_1} + \frac{1}{\rho_2} \right) \right].
\]

Finally, for no radial aberrations in the symmetric case, the condition is

\[
1 + \frac{1}{\rho_1} + \frac{1}{\rho_2} = \frac{2}{R \cot^2 \theta}.
\]

(29)

The case \( \theta = 45^\circ \), \( \rho_1 = \rho_2 = R \) gives the Browne and Buechner magnet.\(^{14}\) The cases \( \rho_1 = \rho_2 \) or \( \rho_1 = \infty \) are mentioned by Bainbridge.\(^{15}\)

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**APPENDIX I**

**GENERAL REMARKS ON THE FRINGING FIELD OF A MAGNET**

Let us consider the trajectory in the symmetry plane. It is usual to consider the "ideal magnet" with a discontinuous field such that the integral \( \int B dt \) is the same for


From identity (25), one may deduce
\[
\int_{\text{inside the magnet}} \left[ \frac{S_r'^2(1-hD)-2hSS'}{p_0} \right] dt = \frac{-1}{k} \left[ S_r'^2 D' \right]_{\text{exit}}^{\text{entrance}},
\]
\[
\left( \frac{r}{r_0} \frac{\Delta p}{p_0} \right)_{\text{exit}} = -MR \left[ S_r'^2 D' \right]_{\text{entrance}}^{\text{exit}} - M \int_{\text{outside the magnet}} S_r'^2 dt.
\]
From the matrices expressed, for instance, in Ref. 9 one may calculate \( S_r'^2D' \) at the exit of the magnet = \( \sin \alpha (l_0/R \sin \alpha - \cos \alpha) \) and at the entrance = 0
\[
\int S_r'^2 dt \text{ before the magnet} = l_0
\]
\[
\text{after} = \left( \frac{\cos \alpha - \sin \alpha}{R} \right)^2 l_1,
\]
\[
M = \text{magnification} = \left( \frac{\cos \alpha - \sin \alpha}{R} \right).
\]
Therefore
\[
\left( \frac{r}{r_0} \frac{\Delta p}{p_0} \right) = -\left( \frac{\cos \alpha - \sin \alpha}{R} \right) \left[ l_0 + \left( \frac{\cos \alpha - \sin \alpha}{R} \right)^2 l_1 \right.
\]
\[
+ R \sin \alpha \left( \frac{l_0}{R} \left( \frac{l_0}{R} \sin \alpha + \cos \alpha \right) \right).
\]

The relation between \( l_0 \) and \( l_1 \) is
\[
\left( \frac{l_0}{R} \sin \alpha - \cos \alpha \right) \left( \frac{l_1}{R} \sin \alpha - \cos \alpha \right) = 1.
\]
Finally after some simplifications
\[
\left( \frac{r}{r_0} \frac{\Delta p}{p_0} \right) = R \sin \alpha \left( \frac{l_1}{R} \right. + 1 - \cos \alpha + \frac{l_0}{R} \sin \alpha,
\]
and
\[
\tan \psi = \sin \alpha \left[ 1 - \frac{l_0 l_1}{(1 - \cos \alpha) R^2 + l_0 R \sin \alpha} \right].
\]

This result can also be obtained, and more rapidly in this simple case, by the matrix method developed by Brown et al. But our formalism may be helpful in some more complicated examples, particularly, as pointed out, for instance, by Brown, to determine the influence of some additional lens on the value of the aberration coefficient.

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18 R. Belbeoch, P. Bounin, and K. L. Brown, SLAC Publication 132, Stanford, California (1965), and Ref. 13, p. 141.